Quantum theory of an accelerating universe

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We review some of the well-known features of quantum cosmology, such as the factor ordering problem, the wave function and the density matrix, for a dark energy dominated universe, where analytical solutions can be obtained. For the particular case of the phantom universe, we suggest a quantum system in which the usual notion of locality (non-locality) of quantum information theory have to be extended. In that case, we deal also with a quantum description where the existence of a non-chronal region around the big rip singularity is explicitly accounted for.

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I. INTRODUCTION

Besides confirming that the universe expands in an accelerated fashion, recent data coming from SNIa and other observations [1][2] leave the issue of the precise way in which such an accelerated behavior actually occurs unsettled. In fact, the possibility for a superaccelerated expansion beyond what is predicted by a cosmological constant has been raised, implying serious theoretical difficulties. Several models have been proposed so far in order to explain accelerated expansion. The most popular among them are the so-called quintessence models, which are characterized by a universe filled with an homogeneous fluid with an equation of state $p = w\rho$, where p and ρ are the pressure and the energy density of the fluid, respectively, and w is a constant within the range $-1 < w < -\frac{1}{3}$, being the value of $w = -1$ equivalent to a cosmological constant. This would trigger by itself an exponential expansion if one assumes no other energy or matter sources in the universe. This acceleration would correspond to a value for w less than -1, so allowing for what is called the phantom regime. This regime entails violation of the dominant energy condition, and might imply some interesting features from the point of view of the quantum theory. Some of such features will be analyzed in this paper within the realm of a more general formalism where other possible scenarios are also contemplated.

On the other hand, from a quantum mechanical standpoint, the universe is a rather special system since it cannot be described as a whole in terms of space-time coordinates but in terms of geometries. Hence, it offers a particularly interesting framework to deal with some well-known issues of quantum mechanics, such as those related with the notion of non-locality.

In this paper we shall consider therefore the quantum theory of an accelerating universe which is filled with dark energy, both when the dominant energy condition is satisfied and for vacuum contents where such a condition is manifestly violated. In the latter case the notion of non-locality is discussed in a multiverse scenario, where it must be necessarily generalized or extended. The generalized quantum theory of Hartle [9] is then applied to

cases where the future singularities are replaced for non-chronal bounded regions.

The paper can be outlined as follows. Sec. II deals with a phantom universe covering the entire time interval in such a way that it becomes describable as a multiverse. In sec. III we review the canonical Hamiltonian formalism for a quantum universe, particularizing in the problems related with the density matrix and the possibility for the existence of entangled states in the phantom multiverse. The generalized quantum theory is applied to the quantum multiverse in which the big rip singularity is replaced for a bounded non-chronal region in sec. IV, where a decoherence function is used and observable probabilities are isolated from it using the Hartle procedure. We summarize and conclude in sec. V. An appendix on the orthogonality properties of the wave function is also added.

II. THE PHANTOM MULTIVERSE

For a flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe filled only with dark energy, the equations of motion can be obtained from the Hamiltonian constraint in the phase space,

$$\mathcal{H} = -\frac{2\pi G}{3}\frac{p_a^2}{a} + \rho_0 a^{-3w} = 0, \qquad (1)$$

where p_a the canonical momentum, G is the gravitational constant, ρ_0 is the energy density at the coincidence time, i.e., the time in which the dark energy started to dominate the expansion of the universe, and a is measured in units of a_0. In deriving Eq. (1) we have used the integrated form of the energy conservation law,

$$d\rho = -3(p + \rho)\frac{da}{a}. \qquad (2)$$

In the configuration space, the corresponding Friedmann equation reads

$$-\frac{3}{8\pi G}a\dot{a}^2 + \rho_0 a^{-3w} = 0, \qquad (3)$$

and hence the scale factor runs as

$$a(t) \propto (t_{br} \pm t)^{-\frac{2}{3(|w|-1)}} \qquad (4)$$

where t_{br} is a constant, and the $+$ and $-$ signs stand for the quintessence and phantom regimes, respectively. The energy density goes then as

$$\rho \propto a^{3(|w|-1)} \qquad (5)$$

so, in the phantom case, t_{br} turns out to be the time at the so-called big rip singularity [3], where the scale factor and the energy density blow up to infinity (see Fig. 1).

If one had to look at this phantom universe as evolving from an initial coincidence time to infinity, then in order for the scale factor, a, to be positive in (4), not all values of the parameter w entering the equation of state for the universe would be allowed. Actually, for a phantom regime only the values given by [4]

$$w = -\frac{1}{3}\left(1 + \frac{2n+3}{n+1}\right), \quad n = 0, 1, 2, ...\infty, \qquad (6)$$

are allowed as the scale factor becomes ill-defined at $t > t_{br}$ otherwise. Moreover, for observers staying at times before the big rip the singular character of this would necessarily imply that the big rip should be cut off from the considered physical manifold, so making completely unphysical the region beyond the big rip singularity. Thus, besides introducing the condition (6) one must also take into account either that stable wormholes and ringholes (whose existence is induced by the violation of the dominant energy condition implied by the phantom nature of the cosmic fluid) would crop up at both sides of the singularity shortcutting the space-time [5], so keeping the singularity outside the trajectories followed by physical signalling and making therefore accessible to any observers the contracting region beyond the big rip, no matter whether it is cut off or not; or even by considering that quantum effects could somehow smooth out the singular character of the big rip. The condition (6) could be thus interpreted as a condition for the existence of a multiversal scenario in which every value of n would provide us with a different universe with its own time evolution for the scale factor. It is worth considering that this multiversal scenario can still be a classical one, though its quantum counterpart presents some interesting features that we will discuss later on.

III. THE QUANTUM DARK ENERGY UNIVERSE

A. The wave function

Following the canonical quantization procedure, the dark energy dominated universe would be quantum-mechanically characterized, in the configuration space, by a wave function, $\Phi(a)$, which is annihilated by the
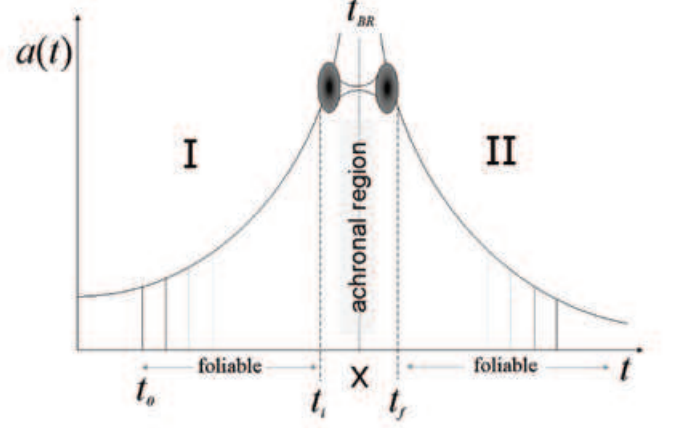

FIG. 1: Wormholes could connect both sides of the universe avoiding that way the big rip singularity. For times t, $t_i < t < t_f$, i.e., the achronal region, the manifold cannot be foliated.

Hamiltonian density (1), in its operator form, satisfying in this way the quantum version of the Hamiltonian constraint. But in doing that we have to make a particular choice of the factor ordering between the conjugate variables. There are many ways in which one can depict this ambiguity, maybe a quite general one would be the following. Classically, the conjugate variables commute so the rhs and lhs in the following expression are equivalent

$$\frac{p_a^2}{a} \equiv a^{-(r+s+1)} p_a a^r p_a a^s, \qquad (7)$$

but, of course, their quantum counterparts are not necessarily equivalent as they depend on the commutation relations between the conjugate variables. For instance, for the kinetic term of the Hamiltonian (1), we would obtain

$$\frac{\hat{p}_a^2}{\hat{a}} \rightarrow \frac{1}{\hat{a}}\hat{p}_a^2 - \frac{(1-2\alpha)[\hat{a},\hat{p}_a]}{\hat{a}^2}\hat{p}_a + \frac{\beta^2[\hat{a},\hat{p}_a]^2}{\hat{a}^3}, \qquad (8)$$

where we have used that

$$[a^l, p_a] = l[a, p_a]a^{l-1}, \qquad (9)$$

and α and β are related to the exponents in Eq. (7) through

$$s = -\alpha \pm \sqrt{\alpha^2 - \beta^2}, \qquad (10)$$

$$r = 1 \mp \sqrt{\alpha^2 - \beta^2}, \qquad (11)$$

so under canonical quantization, $p_a \rightarrow -i\hbar\frac{\partial}{\partial a}$, α and β represent the factor ordering ambiguity in the corresponding Wheeler-DeWitt equation [6], which for the case being considered is

$$\mathcal{N}\left(\frac{1}{a}\partial_{aa}^2 + \frac{1-2\alpha}{a^2}\partial_a + \frac{\beta^2}{a^3} + \frac{\lambda_0^2}{\hbar^2}a^{-3w}\right)\Phi(a) = 0, \quad (12)$$

where $\lambda_0^2 = \frac{3}{2\pi G}\rho_0$, and $\mathcal{N}$ is the lapse function. Here, as well as in Eq. (1), we have assumed: (i) that dark energy dominates in such a way that all matter in the universe is subdominant and can therefore be disregarded, and (ii) that the quantum state is given as a pure state. The first assumption stems from the feature that the 70% of the universal energy is made up of dark energy, even though other fields should be considered in a more detailed framework. The second assumption can be thought of as a plausible one in spite of the state derived from it is not the most general. The solutions of the Wheeler-DeWitt equation, Eq. (12), can be given in terms of Bessel functions

$$\Phi(a) = a^\alpha \, \mathcal{C}_\nu(\lambda a^q), \tag{13}$$

where

$$q = \frac{3}{2}(1 - w), \; \lambda = \frac{\lambda_0}{\hbar q}, \; \nu^2 q^2 = \alpha^2 - \beta^2, \tag{14}$$

$\mathcal{C}$ is the Bessel function of the first or second kind, $\mathcal{J}_\nu$ and $\mathcal{Y}_\nu$, respectively, and w satisfies the discretization (6) for the case of the phantom multiverse, and where we have kept the Planck constant. This wave function would quantum-mechanically describe a quintessence energy dominated universe, and actually the state of the quantum universe for the phantom regime, were it not for the possible existence of a noncausal multiply connected region around the big rip singularity in the phantom energy dominated universe, at least for the fraction of physical reality which always be outside the connection. That wave function must satisfy, as usual, given boundary conditions. For these boundary conditions we choose: (i) the wave function ought to be regular everywhere, even when the metric degenerates at time $t \to \infty$ in the phantom universe, and (ii) it should vanish at the big rip singularity when $a \to \infty$. These conditions are satisfied by the wave function (13) if we enforce α to be

$$\alpha < \frac{q}{2}, \tag{15}$$

for the quintessence regime for $w > -1$ and for $w < -1$ just in the region before the big rip, t_{br}. In the latter case, for times after the singularity we must add the condition

$$\alpha \pm q \, Re(\nu) > 0, \tag{16}$$

where the $+$ sign stands for the Bessel function of the first kind, $\mathcal{J}_\nu$, and the $-$ sign does for the Bessel function of the second kind, $\mathcal{Y}_\nu$. A linear combination of these two solutions together with the boundary conditions (15) and (16) would represent, therefore, the quantum state of a dark energy dominated universe. From this state we should be able to recover the semi-classical universe in which we live. This may be accomplished by taking the limit $\hbar \to 0$ in the expression for the wave function. In particular, using the asymptotic expansion of the Bessel functions [7] for the wave function (13), we can obtain

that in the semiclassical approximation

$$\Phi \sim \sqrt{\frac{2}{\pi\lambda_0}} a^{(\alpha - \frac{1}{2}q)} e^{\pm i(\lambda \, a^q - \frac{1}{2}\nu\pi - \frac{1}{4}\pi)}. \tag{17}$$

This wave function represents the state of the classical universe in the sense that it is a quasi-oscillatory wave function whose argument is essentially the classical action $(S_0 = \lambda a^q)$, so that the correlations between the classical variables [8] are satisfied, i.e, $p_a = \frac{\partial S_0}{\partial a}$, where p_a is the classical momentum, is the equation of motion, and

$$\Delta = a^{\alpha - \frac{1}{2}q} \tag{18}$$

in Eq. (17) is a prefactor smooth enough to satisfy the Hartle criterion [9]. In fact,

$$\mathcal{G}_{ijkl} \frac{\delta}{\delta h_{ij}} (|\Delta|^2 \frac{\delta S_0}{\delta h_{kl}}) = 0, \tag{19}$$

which in our case implies

$$\frac{1}{a} \frac{\partial}{\partial a} \left(a^{2\alpha - q} \frac{\partial S_0}{\partial a} \right) \sim a^{2\alpha - 3} \to 0, \tag{20}$$

that is, the Hartle criterion is satisfied for quintessence models $(w > -1)$ for the boundary condition (15). For the phantom regime we should replace $2\alpha < q$ for $2\alpha < 3$ for the boundary condition, so that Eqs. (15) and (20) are both satisfied. In both cases, the semiclassical approximation (17) obeys the Hamilton-Jacobi equation in the limit $\hbar \to 0$, irrespective of the value of q in the phantom multiverse that is described by (6), and for all choices of factor ordering satisfying the boundary conditions. In particular, any universe in this phantom multiverse would have a semi-classical domain described by (17).

B. The density matrix

The most general quantum state would be given however in terms of a mixed density matrix rather than a pure wave function [10]. One can also compute the density matrix for the case being considered by taking

$$\rho(a', a) = \int_0^\infty dT \, K(a', T; a, 0), \tag{21}$$

where $K(a', T; a, 0)$ is the Schrödinger propagator and the integration over time is introduced to account for the invariance of the time separation between any two hypersurfaces.

In the case of our dark energy universe, we can take the gauge $\mathcal{N} = a^3$ in Eq. (12), so we get a set of Hamiltonian eigenfunctions

$$\hat{H}\Phi_k(a) = \beta_k^2 \Phi_k(a) \tag{22}$$

given by

$$\Phi_k(a) = N_k \, a^\alpha \, \mathcal{J}_k(\lambda a^q), \tag{23}$$

where N_k is a normalization factor, and q and λ are given by Eq. (14), with eigenvalues

$$\beta_k^2 = q^2 k^2 - \epsilon_0^2, \qquad (24)$$

where $\epsilon_0^2 = \alpha^2 - \beta^2 \geq 0$ for the parameters r and s in Eq. (7) to be real (see Eqs. (10) and (11)).

However, this set of eigenfunctions are not orthogonal. For instance, choosing for the scalar product

$$\langle f|g \rangle = \int_0^\infty da\, W(a)\, f(a)g(a), \qquad (25)$$

weighted by the function $W(a) = a^{-(2\alpha+1)}$, the set of eigenfunctions would satisfy the following normalization relations

$$\langle \Phi_k(a)|\Phi_k(a) \rangle = 1, \forall k > 0, \qquad (26)$$

in which we have used in Eq. (23) $N_k = \sqrt{2qk}$; In this way, for $k \neq l$, we obtain

$$\langle \Phi_k(a)|\Phi_l(a) \rangle = 0 \qquad\qquad , (k-l) \quad \text{even}$$

$$\langle \Phi_k(a)|\Phi_l(a) \rangle = \frac{4}{\pi}\frac{(-1)^{\frac{1}{2}(k-l-1)}\sqrt{kl}}{k^2-l^2} \quad , (k-l) \quad \text{odd}. \qquad (27)$$

This set of equations can be considered as orthogonality relations in the sense that they permit to split the whole Hilbert space spanned by the Hamiltonian eigenfunctions into two Hilbert subspaces, i.e., the subspaces spanned by the odd and even modes. In that case, the Hamiltonian eigenfunctions form two orthogonal basis, in the usual sense, for the subspaces.

But the zero mode is not normalizable. We may regularize it by using some cut-off or minimum length, l_p, taking on the limit $l_p \to 0$ at the end of the calculations. We have (see the Appendix),

$$\langle \Phi_0|\Phi_0 \rangle = \lim_{k,l\to 0}\langle \Phi_k|\Phi_l \rangle \sim \frac{N_0^2}{q}\ln(\frac{2}{\lambda l_p^q}) + \mathcal{O}(k \pm l), \quad (28)$$

with which we could take the normalization relations (26) for all $k \geq 0$, with a suitable normalization factor, N_0.

With that regularization, we can make use of the following set of functions

$$\Psi_n(a) = \sqrt{q\lambda}\, a^{\alpha+\frac{1}{2}q}\, e^{-\frac{\lambda a^q}{2}} L_n(\lambda a^q), \qquad (29)$$

which are constructed in terms of the Laguerre polynomials, $L_n(x)$,

$$L_n(x) = \sum_{m=0}^n \binom{n}{m}\frac{(-x)^m}{m!}. \qquad (30)$$

We have in this way obtained an orthonormal set under the scalar product (25), which can be used as the basis for the square integrable functions upon which the Hamiltonian would act, so

$$\sum_n |\Psi_n\rangle\langle \Psi_n| = Id. \qquad (31)$$

Using this set, we have

$$Id = \sum_n |\Psi_n\rangle\langle \Psi_n| = \sum_{kl} \mathcal{D}_{kl}|\Phi_k\rangle\langle \Phi_k| \qquad (32)$$

where

$$D_{kl} = \sum_n C_{nk}C_{nl}^*, \qquad (33)$$

and the C_{ij}'s are the coefficients for the change of basis, i.e.,

$$\Psi_n(a) = \sum_m C_{nm}\Phi_m(a). \qquad (34)$$

Hence, the propagator can be written as

$$\begin{aligned} K(a',T;a,0) &= \sum_{kl} D_{kl}\langle a'|e^{\frac{i}{\hbar}TH}|\Phi_k\rangle\langle \Phi_l|a\rangle \\ &= \sum_{kl} D_{kl}e^{\frac{i}{\hbar}T\beta_k^2}\Phi_k(a')\Phi_l^*(a), \quad (35) \end{aligned}$$

and the density matrix computed to be

$$\rho(a',a) = \sum_{kl} D_{kl}\frac{\Phi_k(a')\Phi_l(a)}{q^2k^2-\epsilon_0^2}, \qquad (36)$$

where, in order to make the integral well-defined, we have Wick rotated time counterclockwise. Wick rotating in the opposite direction would have implied inserting the identity before the evolution operator and setting a minus sign in the exponent.

In the case of the dark energy universe, with the Hamiltonian eigenfunctions (23), the coefficients in Eqs. (33) and (34) can be computed. For, we can take advantage of the properties of the first kind Bessel functions, which form up an overcomplete set in the sense that any arbitrary function can be decomposed in terms of Bessel functions through a Neumann's expansion [11]

$$z^\nu f_k(z) = \sum_{n=0}^\infty c_{kn}\mathcal{J}_{n+\nu}(z), \qquad (37)$$

with the coefficients being given by

$$c_{kn} = \frac{1}{2\pi i}\int_{|t|<R} dt\, f_k(t)A_{n,\nu}(t), \qquad (38)$$

where R is the distance from $t = 0$ to the closest pole of $f_k(t)$, and the $A_{n,\nu}$, the Gegenbauer's polynomials, are defined by

$$A_{n,\nu}(t) = \frac{2^{n+\nu}(n+\nu)}{t^{n+1}}\sum_{m=0}^{\leq\frac{1}{2}n}\frac{\Gamma(n+\nu-m)}{m!}\left(\frac{t}{2}\right)^{2m}. \qquad (39)$$

Thus, we can rearrange Eq. (37) so that any arbitrary function could be written as

$$g_k(a) = a^\alpha (\lambda a^q)^\nu f_k(\lambda a^q) = \sum_{n=0}^\infty c_{kn}a^\alpha \mathcal{J}_{n+\nu}(\lambda a^q). \quad (40)$$

In particular, with $\nu = \frac{1}{2}$, the orthonormal set of functions (29) can be decomposed as

$$\Psi_k(a) = \sum_{n=0}^{\infty} C_{kn} \Phi_{n+\frac{1}{2}}(a) \qquad (41)$$

in which the coefficients, unless for a normalization constant, are given by Eq. (38), with

$$f_k(t) = \sqrt{q}\, e^{-\frac{t}{2}} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^l}{l!} t^l, \qquad (42)$$

that is,

$$
\begin{aligned}
C_{kn} &= \frac{(n+\frac{1}{2})\, 2^{n+\frac{1}{2}}}{N_{n+\frac{1}{2}}} \frac{1}{2\pi i} \sqrt{q} \sum_{l=0}^{k} \sum_{m=0}^{\le \frac{1}{2}n} \binom{k}{l} \frac{(-1)^l\, \Gamma(n+\frac{1}{2}-m)}{2^{2m}\, l!\, m!} \int_{|t|<R} \frac{e^{-\frac{t}{2}}}{t^{n-2m-l+1}} dt \\
&= \frac{(-1)^n\, (n+\frac{1}{2})\, 2^{n+\frac{1}{2}}}{N_{n+\frac{1}{2}}} \sqrt{q} \sum_{m \ge \frac{n-k}{2}}^{\le \frac{1}{2}n} \binom{k}{n-2m} \frac{\Gamma(n+\frac{1}{2}-m)}{2^{2m}\,(n-2m)!\, m!},
\end{aligned}
\qquad (43)
$$

where m is a non-negative integer. The density matrix (36) can then be written as

$$\rho(a', a) = \sum_{kl} D_{kl} \frac{\Phi_{k+\frac{1}{2}}(a')\Phi_{l+\frac{1}{2}}(a)}{q^2(k+\frac{1}{2})^2 - \epsilon_0^2}. \qquad (44)$$

A density matrix for a physical system is supposed to be definite positive, and Eq. (44) is however not necessary so, even for positive values of the coefficients D_{kl}. Parameters α and β have nevertheless no clear physical meaning since a semiclassical state should be independent of the particular choice of α and β. We then could still take for the physical state of the universe the reduced density matrix resulting from integrating out α and β, i.e.,

$$\rho_r(a', a) = \int d\alpha \int d\beta\, \rho(a', a; \alpha, \beta), \qquad (45)$$

This integral turns out to be divergent so that one could not obtain a meaningful density matrix for the state of the universe. One way to avoid this problem could be taking some particular values for the factor ordering as a boundary condition. For the particular choice $\alpha = \beta = 0$ we would in fact have

$$\rho(a', a) = \sum_{nm} D_{nm} \frac{\Phi_{n+\frac{1}{2}}(a')\Phi_{m+\frac{1}{2}}(a)}{q^2(m+\frac{1}{2})^2}, \qquad (46)$$

where the coefficients are given by Eq. (33). However, even though this density matrix does not show the usual divergences due to vanishing values of the denominator, the coefficients D_{nm} in it are still divergent.

On the other hand, our particular choice leading to the density matrix (46) corresponds to a rather arbitrary choice of the factor ordering. Other choices could also imply a definite positive density matrix. It would follow that an alternate philosophy could be constructing a propagator in terms of pairs of levels instead of the single levels which correspond to the Hamiltonian eigenvalues, that is

$$\langle a'|e^{iTH}|a\rangle = \sum_{nm} D_{nm} \langle a'|e^{i\frac{TH}{2}}|\Phi_{n+\frac{1}{2}}\rangle\langle\Phi_{m+\frac{1}{2}}|e^{i\frac{TH}{2}}|a\rangle, \qquad (47)$$

In that case, we have

$$\rho(a', a) = \sum_{nm} D_{nm} \frac{\Phi_{n+\frac{1}{2}}(a')\Phi_{m+\frac{1}{2}}(a)}{q^2(m^2 - n^2 + m - n)}, \qquad (48)$$

i.e., although the factor ordering ambiguity is no longer present in the denominator, the diagonal elements become now divergent.

The kinds of divergences and unphysical states that we have just uncovered were already pointed out by Hawking and Page for density matrix in quantum cosmology [10] [12]. Actually, these difficulties can be seen to arise because the system can reach a state with zero value for its Hamiltonian eigenvalue, and would be expected to be solved in the framework of a proper quantum theory of gravity, in which a minimum energy, the Planck mass, m_p, should exist. In such a case, a nonzero minimum value of the Hamiltonian eigenvalue would be expected that rendered the density matrix definite positive and always convergent.

C. Entangled states in the multiverse

Let us, now, be concerned with a phantom universe and its big rip singularity, then we could consider the case in which no wormholes are connecting the regions before and after the big rip; i.e. when the wormholes that branch off in the neighborhood of that singularity simply connect two asymptotic regions on the same side of the singularity. If thereby such wormholes are disregarded and the hypersurface at the singularity is cut out

so that the whole space-time is divided into two separate parts, then there will be two independent wave functions which should be associated with different realizations of the boundary conditions and distinct time intervals. The first of these intervals runs from the coincidence time until the time at the big rip, and the second one goes from the latter time until infinity. The general boundary conditions that the quantum state be regular everywhere and exactly vanishes at the big rip singularity amount to a wave function which should be generally expressed in terms of a different linear combinations of first and second kind Bessel's functions, $\mathcal{J}$ and $\mathcal{Y}$, on each interval. These two wave functions for both sectors can be regarded to play the role of some bases for the quantum state of a specific n-phantom universe. So, in general we can describe this state as

$$\Psi_n = c_I^n \Psi_I^n + c_{II}^n \Psi_{II}^n, \tag{49}$$

with

$$(c_I^n)^2 + (c_{II}^n)^2 = 1. \tag{50}$$

Let us interpret for a moment the integer number n defined in Eq. (6) as a quantum number labeling the different universes in our multiverse, and then consider the quantum states for two universes with different values of the quantum number n. In that case, ψ_I^n and ψ_{II}^n, which quantum-mechanically describe the regions before and after the singularity for a single universe labeled n, are both strongly peaked at time-like separated regions and can therefore be correlated to the similar regions of the other universe, say m, as all of such states satisfy the so called Hartle criterion (20). The result for the two universes could be a common state

$$\Phi = c_I^n c_I^m \Psi_I^n \Psi_I^m + c_{II}^n c_{II}^m \Psi_{II}^n \Psi_{II}^m. \tag{51}$$

Since the singularity has been cut off, mixed states $\psi_I^n \psi_{II}^m$ and $\psi_{II}^n \psi_I^m$ are no longer possible if there are correlations between $\psi_I^n \psi_I^m$ and $\psi_{II}^n \psi_{II}^m$ and, therefore, state (51) should be an entangled state.

Eq. (51) can straightforwardly be generalized to the infinite possible number of universes and would imply that knowing the state of our universe we would automatically know the state of the other universes belonging to the same multiverse scenario. This is a cosmic translation from what is usually dubbed quantum non-locality in quantum information theory. Since there is no space-time between any two universes in this multiverse, the term locality (or non-locality) used to characterize correlations between two particles in a common space-time, is no longer suitable to physically characterize the above correlations between universes, provided that locality refers to just space-like or time-like location in a common space-time. It would instead refer to correlations between the quantum states of different universes, which would become entangled as a result.

IV. GENERALIZED PHANTOM UNIVERSE

However, the existence of a non-chronal region around the big rip makes it impossible to have any quantum state for the phantom universe. Actually, in order for having a proper quantum theory of one of the universes of the multiverse we have to make use of a generalized quantum theory [9]. Technically, the physical system we ought to deal with consists of a space-time manifold containing an intermediate bounded non-foliating region on the neighborhood of the big rip singularity, filled with closed time-like curves (CTCs), which is chronologically placed between two regions that are both foliable by a family of nonintersecting spacelike surfaces Σ_p (See Fig. 1). We then introduce the generalized decoherence function of Hamiltonian mechanics which reads

$$D(\alpha', \alpha) = N\text{Tr}\left[P_{\alpha_p'}^p(\Sigma_p)...P_{\alpha_{k+1}'}^{k+1}(\Sigma_{k+1}) X P_{\alpha_k'}^k(\Sigma_k)...P_{\alpha_1'}^1(\Sigma_1) \rho P_{\alpha_1}^1(\Sigma_1)...P_{\alpha_k}^k(\Sigma_k) X^\dagger P_{\alpha_{k+1}}^{k+1}(\Sigma_{k+1})...P_{\alpha_p}^p(\Sigma_p) \right], \tag{52}$$

where the P_α's are projection operators which forms up a set, $\{P_{\alpha_p}\}$, that corresponds to the exhaustive and exclusive set of alternatives defined on a given non-intersecting spacelike surface Σ, and the α_j's are particular sequences of coarse-grained alternatives $\{\alpha\} = \alpha_1, ..., \alpha_j$ that describe particular histories. The exhaustive set of histories consists then all possible sequences $\{\alpha\}$. The function ρ denotes the density matrix encompassing the boundary condition of the system on an initial nonintersecting spacelike surface Σ_0. It will be given either by the factorizable probability $\rho = W = \Phi(a)\Phi(a')$ or by the expressions for the density matrix described in the previous section, if the probability function W is not factorizable,

i.e., for a mixed state. X is a generalized evolution matrix that can be defined in terms of a nonunitary matrix X_s which replaces the usual unitary evolution matrix U of the decoherence function for fully foliable manifolds. It can be given by

$$X = U\left(\Sigma_f, \Sigma_\infty\right)^{-1} X_s U(\Sigma_i, \Sigma_0), \tag{53}$$

in which Σ_0 and Σ_∞ are surfaces at the furthest possible past and future, the latter being assumed to be at the event horizon (i.e. in the present case at infinity), and Σ_i and Σ_f are the latest and earliest nonintersecting spacelike surfaces, after and before the nonfoliable region, respectively. Finally, the normalizing factor N is

given by

$$N = \left[\mathrm{Tr}(X\rho X^\dagger)\right]^{-1}. \tag{54}$$

It can be then shown that the generalized decoherence function (52) is normalizable and hermitian, has positive diagonal elements, and satisfies the superposition principle, provided that [13]

$$D(\bar{\alpha}', \bar{\alpha}) = \sum_{\alpha' \in \bar{\alpha}'} \sum_{\alpha \in \bar{\alpha}} D(\alpha', \alpha)$$

for all coarse grainings $\{\bar{\alpha}\}$ of $\{\alpha\}$. Thus, this function satisfies all consistency tests.

In the sum-over-histories formulation of gravitational systems the usual probability function W is replaced for a probability function for a given set of alternatives α, $p(\alpha)$, which can be obtained from the decoherence function by using the relation $D(\alpha', \alpha) \approx \delta_{\alpha'\alpha}p(\alpha)$. We shall take these probabilities $p(\alpha)$ as the physical quantities that replace quantum states in our non-causal system. So, we are here particularly interested in calculating from Eq. (52) the probability $p(\alpha, \Sigma; t_0, a)$ of a set of alternatives for particular values of t_0, a and $P_\alpha\left[B(t_0, a)\right]$, that distinguish only the scale factor values on the whole pieces of surfaces , $B(t_0, a)$, which should be spacelike separated from the nonfoliating region. The most general expression for the probability $p(t_0, a)$ (in which we have specialized at the particular slicings $t = t_0$ and $a = a$) that can be obtained from the decoherence function (52) is

$$
\begin{aligned}
p\left(\alpha, \Sigma'; t_0, a\right) &\equiv p\left(\alpha, \Sigma''; t_0, a\right) \\
&= N\mathrm{Tr}\left\{X P_\alpha\left[B(t_0, a)\right]\rho P_\alpha\left[B(t_0, a)\right] X^\dagger\right\} \equiv N\mathrm{Tr}\left\{P_\alpha\left[B(t_0, a)\right] X\rho X^\dagger P_\alpha\left[B(t_0, a)\right]\right\}.
\end{aligned}
\tag{55}
$$

Finally, the quantity

$$p(\alpha, \Sigma') \equiv p(\alpha, \Sigma'') = \int_0^\infty dt_0 \int_0^\infty da\, p\left(\alpha, \Sigma^{(\upsilon)}; t_0, a\right), \tag{56}$$

where the superscript (υ) denotes either $'$ or $''$, is the most general probability for the set of all alternatives that are able to distinguish the scale factor values on spacelike sections that are spacelike separated from the identified nonfoliating region. It will be here regarded to be the quantity that replaces the quantum state for a parallel phantom universe in Hamiltonian quantum cosmology.

Because of the non-unitarity of the evolution in the neighborhood of the big rip singularity, described in Eq. (55) by the non-unitary operator X, the probabilities given by Eq. (56) for alternative histories completely defined on a current local piece located around e.g. our galaxy on a given hypersurface, would depend on the state of the acronal region [13], in the same way as probabilities for retrodiction histories [9] depend on the current and initial states of the universe. In fact, by the cyclic property of the trace, we can rewrite Eq. (55) as

$$p\left(\alpha, \Sigma'; t_0, a\right) = N\mathrm{Tr}\left\{\rho_f P_\alpha\left[B(t_0, a)\right]\rho P_\alpha\left[B(t_0, a)\right]\right\}. \tag{57}$$

where the density matrix, ρ_f, is given by $\rho_f = XX^\dagger$. It would mean that, experiments in local laboratories, say the solar system or the Galaxy, might give different results depending on whether this non-chronal region in our future exists or not. So, at least from a theoretical point of view, it could glimpse the idea of measuring global properties from *local* experiments.

V. CONCLUSIONS

A multiverse scenario can arise in the realm of a phantom energy dominated universe when we consider the complete range of the time interval, smoothing somehow the big rip singularity. That multiverse scenario comes up from the constraint that we need to impose onto the equation of state parameter, in order to obtain well-defined values for the scale factor at times after the singularity.

Analytic solutions can be obtained for the quantum state of a dark energy dominated universe. If the state of the universe is given by a wave function, i.e., if it is a pure state, its quantum representation can be expressed as a linear combination of Bessel functions and, imposing the appropriate boundary conditions, we can recover the semiclassical approximation in the usual way. Nevertheless, the most general quantum state for the universe should be given by a density matrix, which not with understanding suffers from the usual divergence shortcomings. In order to compute an explicit expression for the density matrix in the case being considered, we use a particular gauge in which the Hamiltonian eigenfunctions can be found, and that can be employed as a basis for the space of functions that the Hamiltonian acts upon, although this basis is not orthogonal. Then, we found an orthonormal set in terms of which the Hamiltonian eigenfunctions can be expressed and, thus, several expressions for the density matrix are given.

We show that cosmic entangled states between universes can take place in the realm of the phantom multiverse. We also give a quantum description of a phantom universe when the singularity is replaced for a bounded

non-chronal region. In such a case, the generalized quantum theory is applied and consistent expressions for the probabilities of alternative histories are given.

Although we have succeeded in obtaining a function that replaces the conventional notion of quantum state in a cosmological spacetime endowed with a bounded multiply connected region, the model considered in this paper is not realistic enough for at least the following reason. We have not specifically introduced any matter fields in the model, so that this should at best be considered as an asymptotic idealization.

Now some kinds of quantum communication channels could be conceived which related the different universes that belong to the multiverse. Whether these communications could be implemented physically between advanced civilizations existing in such universes is a matter that requires further consideration.

APPENDIX A: ORTHOGONALITY PROPERTIES OF BESSEL FUNCTIONS

The usual formula for the integrals of Bessel functions can be obtained from the standard bibliography, to be [7]

$$F(z) \equiv \int^z \frac{1}{t} J_\mu(kt) J_\nu(kt)\, dt = -\frac{1}{\mu^2 - \nu^2}\{kz\,[J_{\mu+1}(kt)J_\nu(kt) - J_\mu(kt)J_{\nu+1}(kt)] - (\mu-\nu)J_\mu(kt)J_\nu(kt)\} \quad \text{(A1)}$$

Then, a definite integral over t summing from 0 to ∞ can be thought of as the substraction of the two following limits,

$$\int_0^\infty dt\, \frac{1}{t} J_\mu(kt) J_\nu(kt) = \lim_{z\to\infty} F(z) - \lim_{z\to 0} F(z). \quad \text{(A2)}$$

Let us compute $\lim_{z\to 0} F(z)$ first. In this case, taking the asymptotic limits for the Bessel's functions, we have

$$\begin{aligned}
\lim_{z\to 0} F(z) &\approx -\frac{1}{\mu^2-\nu^2}\Big\{kz\left[\left(\frac{kz}{2}\right)^{\mu+\nu+1}\left(\frac{1}{\Gamma(\mu+2)\Gamma(\nu+1)} - \frac{1}{\Gamma(\mu+1)\Gamma(\nu+2)}\right)\right] \\
&\quad -(\mu-\nu)\left(\frac{kz}{2}\right)^{\mu+\nu}\frac{1}{\Gamma(\mu+1)\Gamma(\nu+1)}\Big\} \\
&= \left(\frac{kz}{2}\right)^{\mu+\nu}\frac{1}{(\mu-\nu)(\mu+\nu)}\left[2\left(\frac{kz}{2}\right)^2\frac{(\mu-\nu)}{\Gamma(\mu+2)\Gamma(\nu+2)} + (\mu-\nu)\frac{1}{\Gamma(\mu+1)\Gamma(\nu+1)}\right] \\
&\approx \frac{1}{\Gamma(\mu+1)\Gamma(\nu+1)}\frac{\left(\frac{kz}{2}\right)^{\mu+\nu}}{\mu+\nu} \to 0\ (\mu+\nu>0). \quad \text{(A3)}
\end{aligned}$$

We can check that Eq. (A3) vanishes in the limit for any $(\mu+\nu>0)$, but it should be taken into account if a cut off at the Planck length is introduced to regularize the zero mode Bessel function. For the upper limit we obtain

$$\lim_{z \to \infty} F(z) \approx \frac{-1}{\mu^2 - \nu^2} \frac{2k}{\pi} [\cos(z - \frac{1}{2}(\mu+1)\pi - \frac{\pi}{4}) \cos(z - \frac{1}{2}\nu\pi - \frac{\pi}{4})$$
$$- \cos(z - \frac{1}{2}\mu\pi - \frac{\pi}{4}) \cos(z - \frac{1}{2}(\nu+1)\pi - \frac{\pi}{4})] + \mathcal{O}(\frac{1}{z})$$
$$= \frac{k}{\mu + \nu} \frac{\sin(\frac{\pi}{2}(\nu - \mu))}{\frac{\pi(\nu - \mu)}{2}} \tag{A4}$$

For $\mu + \nu > 0$, it gives:

1) if $\mu = \nu$

$$\lim_{z \to \infty} F(z) = \frac{k}{\mu + \nu} \tag{A5}$$

2) if $\mu \neq \nu$ and $\nu - \mu = 2n$ (even)

$$\lim_{z \to \infty} F(z) = 0 \tag{A6}$$

3) if $\mu \neq \nu$ and $\nu - \mu = 2n + 1$ (odd)

$$\lim_{z \to \infty} F(z) = \frac{2k}{\pi} \frac{(-1)^{\frac{1}{2}(\nu - \mu - 1)}}{(\nu - \mu)(\nu + \mu)} \tag{A7}$$

Essentially, these are the values of the integral Eq. (A2) since the limit to zero vanishes. It is left the case for the zero mode. We can regularize it by taking some cut-off or minimum length, and evaluate the limit

$$\lim_{l_p \to 0} \lim_{\mu + \nu \to 0} \int_{l_p}^{\infty} dt \, \frac{1}{t} J_\mu(kt) J_\nu(kt), \tag{A8}$$

expanding in μ and ν the limits Eq. (A3) and Eq. (A4), i.e.

$$\frac{1}{\mu + \nu} \frac{\sin(\frac{\pi}{2}(\nu - \mu))}{\frac{\pi(\nu - \mu)}{2}} \approx \frac{1}{\mu + \nu}(1 + \mathcal{O}((\mu - \nu)^2)) \tag{A9}$$

and

$$\frac{\left(\frac{l_p}{2}\right)^{\mu + \nu}}{\mu + \nu} \approx \frac{1}{\mu + \nu}(1 + (\mu + \nu)\ln(\frac{l_p}{2}) + \mathcal{O}((\mu + \nu)^2)) \tag{A10}$$

so, the limit in Eq. (A8) gives

$$\lim_{\mu + \nu \to 0} \lim_{l_p \to 0} \int_{l_p}^{\infty} dt \, \frac{1}{t} J_\mu(kt) J_\nu(kt) = \lim_{l_p \to 0} \ln(\frac{2}{l_p}). \tag{A11}$$

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